Geometry as a Computational Engine for Continued Fractions of Transcendental Logarithms

Narinder Kumar Wadhawan,
e-mail: narinderkw@gmail.com
Civil Servant, Indian Administrative Service, Now Retired,
Haryana, India

Abstract

The purpose of this paper is to introduce a geometric method using a straightedge and compass for representing the exponent x of an equation, equivalently expressed as x=ln(a)/ln(b), in the form of a continued fraction, thereby enabling its computation. Although analogous to the Euclidean algorithm, this method operates on exponents, with division carried out geometrically rather than symbolically. The exponent of the equation is determined by locating the two perpendiculars between which the magnitude b lies. In the geometric construction, perpendiculars are drawn on the hypotenuse AC and base BC of a right-angled triangle ABC with the right angle at B, where AB=1 and cos(C)=a with a<1. Since the exponent, being transcendental, is not an integer, the process must be repeated for the remainder. The reciprocal of the remainder, treated geometrically, again produces a new remainder, thus continuing the geometric process. This method opens the door to using geometry as a computational tool, rather than restricting it to its traditional illustrative or grammatical role.

1. Introduction

Going back to our school days, when simplifying fractions was part of the curriculum, we encountered many operations—addition, subtraction, multiplication, and division—each requiring careful execution according to a memorised priority rule. This rule was often remembered by the acronym BADMAS, which determined the order of arithmetic operations: first Brackets (BA), followed by Division (D), then Multiplication (M), then Addition (A), and finally Subtraction (S). The combination BA + D + M + A + S formed the word BADMAS (in Hindi बदमाश), which literally means a 'rogue' or 'villain', and this amusing association helped students memorise the order of operations. In contrast, continued fractions involve a single operational priority: computation proceeds from the last term to the first. A simple fraction can be written $a = r/q$ where r and q are real positive integers. But a fraction can also continue as:

$$a = a_0 + \cfrac{a_1}{a_2 + \cfrac{a_3}{a_4 + \cfrac{a_5}{a_6 + \cdots}}}$$

This expansion may involve a finite number of terms (for rational numbers) or an infinite sequence (for irrational or transcendental numbers), where $a, a_0, a_1, a_2,\ldots$ are positive integers. Euclid, in his Elements (c. 300 BCE), introduced an algorithm for computing the greatest common divisor (GCD or HCF) of two numbers [3, 7]. This algorithm forms the backbone of continued fraction construction for a ratio a/b.

Briefly stating, the method finds the GCD of r/q by dividing r by q, yielding quotient a_0 and remainder r_1. Then q is divided by the remainder $r_{1,}$ yielding quotient a_1 and remainder r_2 and the process continues until the remainder vanishes or the division continues indefinitely. The fraction p/q is then expressed as

$$a_0 + \cfrac{a_1}{a_2 + \cfrac{a_3}{a_{4+\frac{a_5}{\ddots}}}}.$$

For example, $375/147$ is written as a continued fraction using the Euclidean algorithm:

$$2+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{4+\cfrac{1}{2+\frac{1}{2}}}}}.$$

In published literature, continued fractions have been visualised geometrically to interpret their properties, their connection to integer lattices, their algorithmic structure, the Farey sequence, and hyperbolic geometry. However, geometry has not been used to extract continued fractions; rather, it has served to analyse the fractions derived from the Euclidean algorithms. These visual interpretations find applications in Diophantine approximation, rational approximation, symmetries, and pattern analysis [5].

Very little work has explored geometry as a *semantic engine for computational purposes*. Foundational contributions in this direction were made by the great geometer René Descartes, who used geometry to solve algebraic problems. In particular, he considered the problem of generating a sequence of lengths between two points a and b such that the ratio of two consecutive terms is constant. To achieve this, he proposed a mechanism involving *movable perpendicular linkages* along the base BC and movable rulers on the hypotenuse AC of a right triangle ABC. This device, now known as *Descartes's Logarithm Machine*, was designed to trace logarithmic curves [1, 2]. The concept was later implemented using dynamic geometry software [1]. More recently, independent semantic constructive approaches, aligned with Descartes' vision, have appeared in published work [8]. The method presented in this paper, developed independently, extends this lineage by using iterative *perpendicular constructions* within a *fixed* triangle to compute the *continued fraction expansion* of the transcendental exponent x in $a^x = b$ —a goal not previously pursued.

In this paper, the geometry using a straightedge and compass is utilised as a computational tool to generate

I. indefinitely continuing fractions of x given by the equation $a^x = b$, where a and b are algebraic and $\neq 0, 1$ or $\pm\infty$ and x is not real rational, and
II. prove the ratio of two transcendental, i.e. $ln(a)/ln(b)$ is transcendental when a and b are algebraic and are $\neq 0, 1$ or $\pm\infty$ and x is not real rational.

1.1 Proof Notations and Definitions

Letters like $A, B, C, \ldots, A', B', C', \ldots$, or $A'', B'', C'', \ldots$ while referring to the geometric Figures 1, 2 and 3, denote points. Two alphabets without gap like $AB, BC, GH, \ldots, A'B', B'C', G'H', \ldots$ $A''B'', B''C'', G''H'', \ldots, D_1D_2, A_3E_4, BD_1 \ldots$ denote a line or its segment. Three alphabets without gap like $ABC, DEF \ldots, A'B'C', D'E'F', \ldots, A''B''C'', D''E''F'', \ldots$, denote a triangle. Geometric signs ⊥, ∠, Δ, denote a perpendicular, an angle, and a triangle, respectively. Alphabet p_m, P_m denote the magnitude of the mth perpendicular corresponding to $cos^m(C)$ and the magnitude of the mth perpendicular corresponding to $1/cos^m(C)$, respectively

Mathematical signs ∞, →, >, <, =, ≥, ≤, denote infinity, tending to (approaching), more than, less than, equal to, equal to or more than, equal to or less than, respectively. Letters $a, b, c, \ldots, x, y, z, \ldots$, denote real quantities. Real quantities r_i, q_i where $i = 0, 1, 2, 3, \ldots$ denote positive integers. $Cos\ (C)$ is the trigonometric ratio of the base to the hypotenuse of a right-angled triangle that has angle C (in radians) opposite to the angle $\pi/2$.

2. Construction and Operation

2.1. Construction of The Right-Angled Triangle ABC with ∠C=a radian

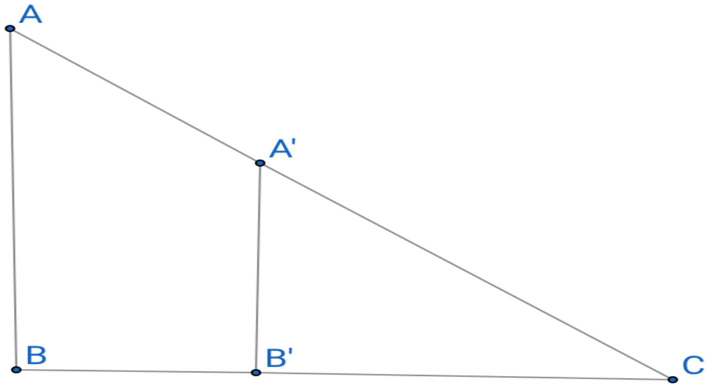

Figure 1 Construction of the right-angled triangle ABC with a line segment $AB = 1$, and ∠C= a radians

When $0 < a < 1$, draw a horizontal line $B'C = a$ unit and construct a perpendicular $A'B'$. With the compass centre at C and opening it equal to 1 unit, draw an arc intersecting A'B' at A' so that CA'=1 unit. Extend CA' to A such that segment AB, perpendicular to CB', meets it at B and equals 1 unit. Now the right-angled Δ ABC has cos(C)= a unit, ∠ABC=π/2 and perpendicular segment AB=1 unit.

When $a > 1$, write the equation $(1/a)^x = 1/b$, and construct the right-angled Δ ABC, assuming a as $1/a$ and following the steps as already explained.

2.2. Construction of $(a)^x = b$

For extracting x in the equation $a^x = b$, equivalently $x = ln(b)/ln(a)$, a and b must be nonzero positive real quantities. If a and b both are less than 1, the right-angled triangle for extracting x is constructible. If a and b both are more than 1; the equation can be written as $(1/a)^x = 1/b$ and the right-angled triangle for extracting x is constructible. If $a < 1$ and $b > 1$, then x will be negative from $ln(b)/ln(a)$, and the substitution $x = -X$ and $B = 1/b$ transforms the equation to $a^X = B$. Similarly, if $a > 1$ and $b < 1$. Then the substitution $A = 1/a$ and $X = -x$ transforms the equation to $A^X = b$. In both cases, the right-angled triangle for extracting X $(-x)$ is constructible.

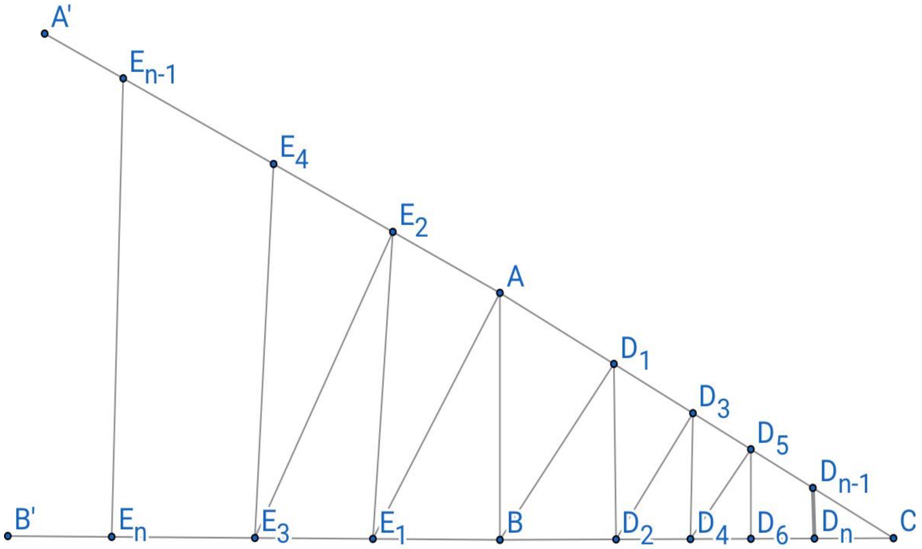

Figure 2 Displaying the construction of $cos^{-n}\ (C)$ to $cos^{n}\ (C)$

Given: Segment lengths a, b, and a unit segment.
Construction: To express x geometrically using a straightedge and compass as given by the equation $a^x = b$, where given $0 < a < 1$ and both a and b are algebraic numbers and $\neq 0, 1$ or $\pm\infty$ and x is not rational, the geometric construction in Figure 2 is drawn as follows:

i. Construct Δ ABC, with base BC, $\angle ABC = \pi/2$, perpendicular segment $AB = 1$. Let $cos(\angle ACB) = a$ or simply $cos(C) = a$ as explained in section 2.1.

ii. Construct segment $BD_1 \perp$ line AC meeting it at D_1. Construct segment $D_1D_2 \perp$ line BC meeting it at D_2. Construct segment $D_2D_3 \perp$ line AC meeting it at D_3. Continue this alternating construction of perpendiculars on lines BC and AC. Let the final segment be $D_{n-1}D_n \perp$ line BC meeting it at D_n. Denote the perpendicular segments BD_1, D_1D_2, D_2D_3,…, $D_{n-1}D_n$ by $p_1, p_2, p_3, \dots, p_n$.

iii. Similarly, construct a segment $AE_1 \perp$ line CB (extension of line CB) meeting it at E_1. Construct segment $E_1E_2 \perp$ line CA (extension of line CA), meeting it at E_2. Continue constructing perpendiculars alternately on CB' and CA'. Let the final segment be $E_{n-1}E_n$. Denote the perpendicular segment AE_1, E_1E_2, E_2E_3,…, $E_{n-1}E_n$ by $P_1, P_2, P_3, \dots, P_n$.

iv. Let the m_0th perpendicular be such that $p_{m_0} > b$ and the $(m_0 + 1)$th perpendicular satisfies $p_{m_0+1} < b$.

v. A new relation emerges: $(a)^{\frac{r_0}{q_0}} = \frac{b}{p_{m_0}}$ or $\left(\frac{b}{p_{m_0}}\right)^{\frac{q_0}{r_0}} = a$, where r_0, q_0 are rational quantities such that $r_0 < q_0$. Lengths of b and a are given and the length of the perpendicular p_{m_0} can be measured by a compass.

vi. Repeat the construction of Figure 2 using this new triangle Δ A'B'C' with $cos(\angle A'C'B') = (b/p_{m_0})$, $A'B' = 1$, right angle at B'. Apply the same perpendicular-dropping procedure to extract the next quotient m_1.

vii. Let the m_1th perpendicular be such that $p_{m_1} > a$ and the $(m_1 + 1)$th perpendicular satisfies $p_{m_1+1} < a$.

viii. This yields a new relation $\left(\frac{b}{p_{m_0}}\right)^{\frac{r_1}{q_1}} = \frac{a}{p_{m_1}}$ or $\left(\frac{a}{p_{m_1}}\right)^{\frac{q_1}{r_1}} = \left(\frac{b}{p_{m_0}}\right)$ where r_1, q_1 are rational quantities such that $r_1 < q_1$.

ix. Let the m_2th perpendicular be such that $p_{m_2} > \left(\frac{b}{p_{m_0}}\right)$ and the $(m_2 + 1)$th perpendicular satisfies $p_{m_2+1} < \left(\frac{b}{p_{m_2}}\right)$.

x. The process will continue ad infinitum and

$$x = m_0 + \cfrac{1}{m_1 + \cfrac{1}{m_2 + \cfrac{1}{m_3 + \cdots}}}$$

where $m_0, m_2, m_3, \dots$, are positive integers [8].

If $a \to 0$ or $a \to 1$, then the $\angle C \to \pi/2$ or $\angle C \to 0$ and it becomes difficult to draw Δ ABC and perpendiculars on base BC and the hypotenuse. In such cases, write the equation $a^x = b$ as $(a)^{-x} = 1/b$ and locate m_0 so that $P_{m_0+1} > 1/b > P_{m_0}$. Proceed as explained in steps v to x.

2.3. Proof

In Δ ABD_1, $\angle ABD_1 = \angle C$, line segment $AB = 1$, therefore, line segment $BD_1 = \cos(C)$. In

$\Delta BD_1D_2, \angle BD_1D_2 = \angle C$, therefore, line segment $D_1D_2 = BD_1cos(C) = cos^2(C)$. Similarly, line segment $D_2D_3 = cos^3(C)$, line segment $D_3D_4 = cos^4(C), ..$, line segment $D_{n-1}D_n = cos^n(C)$. In the same way, line segment $AE_1 = cos^{-1}(C)$, line segment $E_1E_2 = cos^{-2}(C)$, line segment $E_2D_3 = cos^{-3}(C), ..$, line segment $E_{n-1}E_n = cos^{-n}(C)$.

If we consider cos(C) in our continued fractions, then the length of the first perpendicular BD_1 pertains to power 1 of cos(C), length of second perpendicular D_1D_2 to power 2 of cos(C), length of third perpendicular D_2D_3 to power 3 of cos(C) and in this way, the length of the nth perpendicular $D_{n-1}D_n$ to power n of cos(C).

Similarly, if we consider, $1/cos(C)$ in our continued fractions, then the length of the first perpendicular AE_1 pertains to power 1 of $1/cos(C)$, length of the second perpendicular E_1E_2 to power 2 of 1/cos(C), length of third perpendicular E_2E_3 to power 3 of 1/cos(C) and in this way, length of the nth perpendicular $E_{n-1}E_n$ to power n of 1/cos(C).

If $0 < a < 1$, then according to the construction $a = cos(C)$ otherwise $1/a = cos(C)$. When $p_{m_0} > b$ and $p_{m_0+1} < b$, then b corresponds to the length between p_{m_0} and p_{m_0+1}. In other words, it is a fraction r_0/q_0 more than m_0 where $r_0/q_0 < 1$ so that $x = m_0 + r_0/q_0$ and $a^{m_0+r_0/q_0} = b$ and that yields $a^{r_0/q_0} = b/a^{m_0} = b/p_{m_0}$ or $(b/p_{m_0})^{q_0/p_0} = a$ which is again an equation same in structure as $a^x = b$.

Figure 2 is reconstructed but with $cos(C) = b/p_{m_0}$. For this equation also, there are perpendicular segments such that $p_{m_1} > a$ and $p_{m_1+1} < a$. Now $q_0/r_0 = m_1 + r_1/q_1$ resulting in an equation $(a/p_{m_1})^{q_1/r_1} = b/p_{m_0}$. The process will continue ad infinitum resulting in continued fractions.

2.3 Explanation and Numerical Illustration

In $a^x = b$, let $0 < a < 1$. In geometric language, referring to Figure 2, which shows successive perpendiculars in a triangle ABC with a right angle at B and base BC, $cos(C) = a$ —Which perpendicular has its length equal to b? If x is an integer, which is easily identifiable by comparing the length <u>b</u> with that perpendicular. If x is not an integer, we can always find m_0th and $(m_0 + 1)$the perpendicular such that the length of the xth perpendicular lies between them from the inequalities $p_{m_0} > b$ and $p_{m_0+1} < b$, where p_{m_0} and p_{m_0+1} are lengths of m_0th and $(m_0 + 1)$th perpendiculars. Let r_0/q_0 be such that

$$(a)^{m_0+r_0/q_0} = (b), \tag{2.1}$$

where $r_0/q_0 < 1$ and r_0 and q_0 are positive integers and

$$x = m_0 + \frac{r_0}{q_0}. \tag{2.2}$$

In Equation (2.1), m_0 has already been extracted and r_0/q_0 needs extraction but the perpendicular corresponding to $r_0/q_0 < 1$ does not correspond to a nonzero perpendicular. But q_0/r_0 being more than 1 does correspond to a nonzero perpendicular, hence Equation (2,2) is written

$$x = m_0 + \frac{1}{\frac{q_0}{r_0}}. \tag{2.3}$$

This highlights the continued fraction form leading to the extraction of q_0/r_0 by rewriting Equation (2.1) in exponent q_0/r_0 :

$$f^{\frac{q_0}{r_0}} = g,$$

where $f = b/p_{m_0}$, $g = a$, $p_{m_0} = a^{m_0}$ and values of <u>b</u> and a are given, thus facilitating the construction of a right-angled triangle A′B′C′ with $cos(C') = f = b/p_{m_0}$ (base p_{m_0} and hypotenuse b). Let m_1th and $(m_1 + 1)$th perpendicular such that the magnitude of the (q_0/r_0)th perpendicular lies between them from the inequalities $p_{m_1} > g$ and $p_{m_1+1} < g$, where p_{m_1} and p_{m_1+1} are lengths of m_1th and $(m_1 + 1)$th perpendiculars. Let r_1/q_1 be such that

$$(\mathrm{f})^{\mathrm{m}_1+\mathrm{r}_1/\mathrm{q}_1} = (\mathrm{g}), \tag{2.4}$$

where $r_1/q_1 < 1$ and r_1 and q_1 are positive integers. Thus, the value of an integer m_1 is extracted and the value of $q_1/r_1 > 1$ needs extraction. Consequently, the equation (2.4) takes the form

$$x = m_0 + \cfrac{1}{m_1 + \cfrac{1}{\cfrac{q_1}{r_1}}}.$$

Proceeding in this manner, $m_2, m_3, \dots, m_j$ can be extracted, yielding

$$x = m_0 + \cfrac{1}{m_1 + \cfrac{1}{m_2 + \cfrac{1}{m_3 + \cdots}}},$$

and the larger the value of j, the more precise the result becomes.

2.3.1 Numerical Illustration

Let the given equation be $2^x = 5$. This can be written $(1/2)^x = (1/5)$ so that we may construct a triangle ABC with base BC, $AB = 1$, angle $B = \pi/2$ and $cos(C) = 1/2$. From geometry, x lies between the 2nd and the third perpendicular, and that makes $m_0 = 2$ and $(1/2)^{2+r_0/q_0} = 1/5$ or

$$x = 2 + \frac{r_0}{q_0} = 2 + \cfrac{1}{\cfrac{q_0}{r_0}},$$

since $r_0/q_0 < 1$ and does not correspond to any perpendicular 1, 2, 3,…That needs the extraction of $q_0/r_0 > 1$ and writing $(1/2)^{2+r_0/q_0} = 5$ in terms of the exponent of r_0/q_0 :

$$\left(\frac{1}{5p_2}\right)^{\frac{q_0}{r_0}} = \frac{1}{2},$$

where the magnitude of p_2 is known from the construction. From the comparison of lengths of perpendiculars, it is found that ½ lies between the 3rd and 4th perpendiculars in the triangle with $cos(C^{'}) = 1/5p_2$ and that extracts $m_1 = 3$. That yields

$$x = 2 + \cfrac{1}{3 + \cfrac{1}{\cfrac{q_1}{r_1}}},$$

leading to the equation $(1/5p_2)^{3+r_1/\,q_1} = 1/2$ or

$$\left(\frac{1}{2p_3^{'}}\right)^{q_1/r_1} = \frac{1}{5p_2},$$

where $p_3^{'}$ is the magnitude of the third perpendicular. From the comparison of lengths of perpendiculars, it is found that $1/5p_2$ lies between the 9th and 10th perpendiculars in the triangle with $cos(C^{''}) = 1/5p_2$ and that extracts $m_2 = 9$. That yields

$$x = 2 + \cfrac{1}{3 + \cfrac{1}{9 + \cfrac{1}{\cfrac{q_2}{r_2}}}}.$$

Stopping at the third stage and neglecting r_2/q_2, x calculates as 2.321428571 whereas the actual x is 2.321928095 within .021 percent. That demonstrates the validity of the method.

2.4. Computation and Construction of $e^x = b$ by Geometric Construction

2.4.1. Computation and Construction of Euler Number e

Euler number is transcendental and not constructible by using a straight edge and compass. However, it can be approximated geometrically using the formula e equals, limit $n \to \infty$, $(1 + 1/n)^n$ or $n \to \infty$, $e = \{n/(1+n)\}^{-n}$. Set $cos(C) = n/(n+1)$ a right-angled triangle ABC can be constructed with base $BC = n$ and hypotenuse $AC = n + 1$ with $\angle ACB \to 0$. Although $n \to \infty$, is not practically feasible; an approximation can be obtained by choosing n as large as permitted by the size of the drawing sheet or the precision of the computing device (if implemented in software).

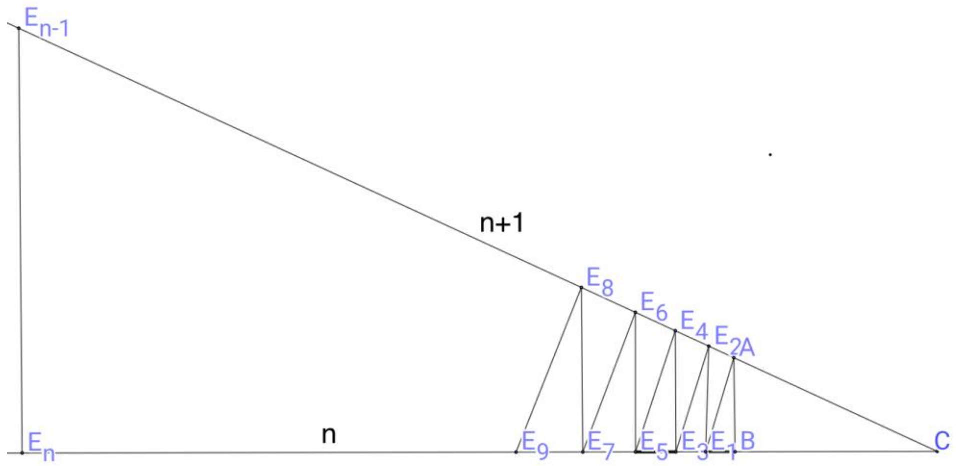

Figure 3 Displaying construction and computation of e

A brief description of the construction is as follows: Referring to Figure 3, construct triangle ABC with $cos(\angle ACB) = n/(n+1)$. From A, drop a perpendicular AE_1 to CB extended, meeting it at E_1. From E_1, drop a perpendicular E_1E_2 to CA extended, meeting it at E_2. From E_2, drop a perpendicular E_2E_3 to CB extended, meeting it at E_3. Continue this process alternately until the nth perpendicular $E_{n-1}E_n$ from E_{n-1} meets CB extended at E_n. From similar triangles ABC, ABE_1, $E_1E_2E_3$ so on, line segment $AE_1 = cos^{-1}(C)$, $E_1E_2 = cos^{-2}(C)$, $E_2E_3 = cos^{-3}(C)$, ..., $E_{n-1}E_n = cos^{-n}(C)$, since line segment $AB = 1$. Therefore, $E_{n-1}E_n = cos^{-n}(C) \approx \{n/(n+1)\}^{-n}$.

2.4.2. Computation and Construction of $e^x = b$

The geometric construction and computation of $e^x = b$ follow the same procedure as that for $a^x = b$ (Sections 2.2 and 2.4), with the following substitutions:

- Replace a by $1/e$,
- Set $cos(\angle ACB) = 1/e$.

Repeating the full details here would duplicate Sections 2.2 and 2.3 in toto and is therefore omitted. For the geometric construction and approximation of Euler's number e, refer to Section 2.4.1 and Figure 3.

3. Comparison With the Euclidean Algorithm

3.1 Euclidean Algorithm

i. The Euclidean algorithm is a general method applicable to fractions r/q where the process of division of q by r is possible and q, r are positive integers.

ii. It divides q by r so that $\frac{q}{r} = m_0 + \frac{r_1}{r} = m_0 + \frac{1}{\frac{r}{r_1}}$. Now r_1 divides r so that $\frac{r}{r_1} = m_1 + \frac{r_2}{r_1} = m_1 + \frac{1}{\frac{r_1}{r_2}}$, thus $\frac{p}{q} = m_0 + \frac{1}{m_1 + \frac{1}{\frac{r_1}{r_2}}}$. Continuing the process, $\frac{q}{r}$ can be written

$$m_0 + \cfrac{1}{m_1 + \cfrac{1}{m_2 + \cfrac{1}{m_3 + \cdots}}},$$

where $m_0, m_2, m_3, \ldots$, are positive integers.

iii. The algorithm's applicability is limited to real rational quantities and polynomials. It can be made applicable to other functions when written in polynomials or in numerical values.

3.2 Geometric Method

i. The geometric method is a special method for continued fraction for obtaining the continued fraction of x in the equation $a^x = b$ or equivalently $x = ln(b)/ln(a)$.

ii. It uses the theory that in a right-angled Δ ABC with base BC, $\angle ABC = \pi/2$, perpendicular segment AB=1 if from point B, a perpendicular BD_1 is drawn on line AC, D_1D_2 on BC, D_2D_3 on AC, then $BD_1 = cos(C), D_1D_2 = cos^2(C),\ D_2D_3 = cos^2(C), \ldots, D_{n-1}D_n = cos^n(C)$. Thus, a perpendicular segment denotes to a specific value of k in $cos^k(C)$.The given value of b can then be found to exist between two perpendicular say m_0 and $m_0 + 1$ where $b = m_0 + r_0/q_0$.

iii. Thus b which is a fraction with integer m_0 (quotient) and as r_0/q_0 remainder. It is tantamount to the division of the numerator by the denominator (of b).

iv. Quotient (m_1) and reminder (r_1/q_1) of fraction q_0/r_0 are extracted geometrically by constructing a right-angled triangle with an angle $\angle C$ corresponding to $cos(C) = b/p_{m_0}$. The process is continued.

v. The geometric method is the same in spirit as the Euclidean algorithm, with the difference that it involves exponents and the division takes place geometrically rather than symbolically.

4. Nature of x in $a^x = b$ and $e^x = b$

***Lemma 4.1**: (Classical corollary of the Gelfond–Schneider theorem): If x in $a^x = b$ or equivalently $x = ln(b)/ln(a)$, is not rational, and $a > 0$ and $b > 0$ are algebraic with $a \neq 1$, and $b \neq 1$. Then x is transcendental.*

Proof: The following proof is a direct application of the classical Gelfond–Schneider theorem (proved independently by A. O. Gelfond in 1934 and T. Schneider in 1935 [4]. If x is not rational, then there are two remaining possibilities, either x is algebraic, irrational, or transcendental.
Case 1: x is algebraic irrational.

By the Gelfond–Schneider theorem, if a is algebraic with a $\neq 0, 1$ and x is algebraic irrational, then a^x is transcendental [4]. But we are given that b is algebraic, so $a^x = b$ cannot be transcendental. This contradiction shows x cannot be algebraically irrational.

Also x is not rational according to the given condition. Therefore, x must be transcendental. This proves lemma 4.1.

Clarifying remark

The status of results obtained from general operations (addition, subtraction, multiplication, division) on two different transcendental numbers cannot be uniformly determined. However, the imposition of specific conditions that x in $a^x = b$ is not rational, and $a > 0$ and $b > 0$ are algebraic with $a \neq 1$, and $b \neq 1$ makes x transcendental as a corollary of Gelfond–Schneider theorem [4].

Lemma 4.2: *Using an unmarked straightedge and compass, let* ΔABC *be a right triangle with* $\angle ABC = \pi/2$, *base BC, and perpendicular AB = 1. From point B , drop a perpendicular* BD_1 *to hypotenuse AC meeting it at* D_1. *Then, from* D_1, *drop a perpendicular* D_1D_2 *to BC meeting it at* D_2 *; from* D_2, *drop a perpendicular* D_2D_3 *to AC meeting it at* D_3*; and continue this process alternately. Then, for any integers* $n > m > 3$ *and* $p > q > 3$, *the ratio of two transcendental logarithms satisfies*

$$\frac{ln\left(\frac{D_{n-1}D_{n-3}}{D_{m-1}D_{m-3}}\right)}{ln\left(\frac{D_{p-1}D_{p-3}}{D_{q-1}D_{q-3}}\right)} = \frac{n-m}{p-q},$$

Proof: Referring to Figure 2, the segment lengths on the hypotenuse AC are given by:

$$D_{n-1}D_{n-3} = D_{n-2}D_{n-3}sin(C) = sin(C)cos^{n-2}(C),$$

$$D_{m-1}D_{n-3} = D_{m-2}D_{m-3}sin(C) = sin(C)cos^{m-2}(C),$$

Thus,

$$\frac{D_{n-1}D_{n-3}}{D_{m-1}D_{n-3}} = cos^{n-m}(C),$$

Taking the natural logarithm yields:

$$n - m = \frac{1}{ln(cos\,C)} ln\left(\frac{D_{n-1}D_{n-3}}{D_{m-1}D_{n-3}}\right). \tag{4.1}$$

$$p - q = \frac{1}{ln(cos\,C)} ln\left(\frac{D_{p-1}D_{p-3}}{D_{m-1}D_{n-3}}\right). \tag{4.2}$$

Dividing (4.1) by (4.2), we obtain

$$\frac{n-m}{p-q} = \frac{ln\left(\frac{D_{n-1}D_{n-3}}{D_{m-1}D_{m-3}}\right)}{ln\left(\frac{D_{p-1}D_{p-3}}{D_{q-1}D_{q-3}}\right)}, \tag{4.3}$$

although the numerator and denominator of the right-hand side of Equation (4.3) are both transcendental. That proves *Lemma 4.2.*

Remark: By Lemma 4.1, when $x = ln(b)/ln(a)$, is not rational, and a > 0 and b > 0 are algebraic with a $\neq 1$, and b $\neq 1$, then x is transcendental. However, in the present construction, the bases are geometrically related via $a = b^k$ for some integer k , due to the uniform scaling by $cos\ C$. This structural constraint forces the ratio of two transcendental logarithms to be rational — a non-trivial consequence of the geometry. *This rational equality arises purely by construction: the iterative perpendicular process imposes algebraic dependence among the bases via uniform scaling by cos C, independent of general transcendence theory.*

Lemma 4.3 *(Classical corollary of the Lindemann–Weierstrass theorem): If algebraic $b > 0$ is a real algebraic number with $b \neq 1$, then x in $e^x = b$, is transcendental.*
Proof:
The following proof is a direct application of the classical Lindemann–Weierstrass theorem (proved by F. Lindemann in 1882 and generalised by K. Weierstrass in 1885): If α is a non-zero algebraic number, then e^{α} is transcendental.

Suppose x is rational, then $b = e^x$ is transcendental, contradicting to the given statement that b is algebraic. Thus, x can not be rational.

Suppose x is algebraic, since x is a non-zero algebraic number, $e^x$is transcendental (by the Lindemann–Weierstrass theorem), again contradicting that b is algebraic. Thus, x cannot be algebraic. The only remaining possibility is that x is transcendental. This proves Lemma 4.3.

5. Convergence and the Rate of Convergence

It is proved in Section 3 that the continued fraction generated by the geometric method is the standard simple continued fraction of the transcendental number $\mathrm{x} = \log_{\mathrm{a}}\mathrm{b}$. Since x is irrational (in fact, transcendental by Lemma 4.1), classical theory guarantees that:

i. The value of the continued fraction generated by the extracted quotients converges to x. (see [6]).
ii. The error after n steps is less than $1/\mathrm{k}_\mathrm{n}^2$, where k_n is the denominator (see [6]).
iii. The denominators grow at least exponentially with n, ensuring rapid convergence (see [6]).

These are well-known properties of continued fractions for any irrational number. The novelty of this work lies not in discovering new convergence behaviour, but in constructing the partial quotients geometrically — using only compass and straightedge within a single right triangle. Where the Euclidean algorithm divides numbers symbolically, this method divides exponential scales geometrically, achieving the same mathematical outcome through pure construction.

6. Results and conclusions

The exponentiation of a real quantity a when $0 < a < 1$ i.e. a^x, for $-\infty < x < +\infty$ can be expressed geometrically in the form of $\cos^{\mathrm{x}}(\mathrm{C})$ using a straightedge and a compass. Construct a right-angled Δ ABC, with base BC, $\angle\mathrm{ABC} = \pi/2$, and perpendicular line segment AB=1. Let $\cos(\mathrm{ACB}) = \mathrm{a}$. Successive perpendiculars are drawn on base BC and hypotenuse AC. Denote $\mathrm{BD_1}, \mathrm{D_1D_2}, \mathrm{D_2D_3}, ..,$ as first, second, third perpendicular... with lengths $\mathrm{p_1}, \mathrm{p_2}, \mathrm{p_3}, ...,$ then $cos^x(C) = a^x = p_x$.

If x is not an integer, suppose it lies between m_0, and $m_0 + 1$ where $\mathrm{m_0}$ is an integer. This can be detected when the magnitude of b in the equation $a^x = b$ satisfies $\mathrm{p_{m_0}} < \mathrm{b} < \mathrm{p_{m_0+1}}$. This identifies $\mathrm{m_0}$. Now consider $x = m_0 + r_0/q_0$, where the fraction $\mathrm{r_0/q_0} < 1$. Then $\mathrm{a^{m_0+r_0/q_0}}$=b or

$$(p_{m_0} a^{r_0/q_0}) = b$$

or

$$\left(b/p_{m_0}\right)^{q_0/r_0} = a.$$

This equation is analogous to $a^x = b$. The integer m_1 in the equation $q_0/r_0 = m_1 + r_1/q_1$ can be found using the same process as for $\mathrm{m_0}$. Similarly, m_2, m_3, m_4 can be extracted, resulting in transcendental

$$x = m_0 + \cfrac{1}{m_1 + \cfrac{1}{m_2 + \cfrac{1}{m_3 + \cdots}}},$$

where $\mathrm{m}_0, \mathrm{m}_1, \mathrm{m}_2, ..$ are positive integers including zero. This continued fraction is infinite and non-terminating [8]. Therefore, the value of the transcendental x can only be approximated by truncating the fraction at a finite number of terms, according to the desired precision. Thus, the geometric construction extracts the terms of the continued fraction and approximates the transcendental value but fails to yield its exact value. Like all other methods, it is a method of approximation.

We demonstrate how approximation is done in [S1]. In summary, for determining x by continued fraction, both a and b are assumed to satisfy $0 < a < 1$ and $0 < b < 1$. The interactive figure displays the values of m_0, m_1 and m_2 for the continued fraction $x = m_0 + \frac{1}{m_1 + \frac{1}{m_2} + \ddots}$, where $x = \frac{ln(b)}{ln(a)}$. Writing $x_0 = m_0, x_1 = m_0 + \frac{1}{m_1}$, $x_2 = m_0 + \frac{1}{m_1 + \frac{1}{m_2}}$, and letting $k_0 = 1$ the denominator of x_0, $k_1 = m_1$ the denominator of x_1, and $k_2 = m_1 m_2 + 1$ the denominator of x_2 and actual $x = \frac{ln(b)}{ln(a)}$, it can be observed numerically from the interactive figure, in agreement with the known bounds for continued fractions, that $|x - x_0| < \frac{1}{k_0^2}$, $|x - x_1| < \frac{1}{k_1^2}$ and $|x - x_2| < \frac{1}{k_2^2}$. It can further be observed that k_n grows exponentially with n, ensuring rapid convergence of the continued-fraction approximation.

The interactive file not only provides continued fractions approximations for $\mathrm{a}^{\mathrm{x}} = \mathrm{b}$ (equivalently $\mathrm{x} = \ln(\mathrm{b})/\ln(\mathrm{a})$, but also stimulates curiosity and a sense of wonder, showing how calculations that normally require logarithmic tables or calculators can be performed geometrically using right-angled triangles and perpendiculars. Observing this simple yet revealing method encourages one to explore whether the same approach can be applied to other problems, thereby planting the seed for further investigation and research.

7. Supplementary Electronic Material

[ICFE] An interactive HTML file.

8. References

[1] Dennis, D., Cinfrey, J.: Deriving logarithmic and exponential curves with the computer software Geometer's Sketchpad: A method inspired by historical sources. In: King, J., Schattschneider, D. (eds.) Geometry Turned On: Dynamic Software in Learning, Teaching and Research, pp. 147–156. Mathematical Association of America, Washington D.C. 1997.

[2] Descartes, R.: The Geometry of René Descartes. Translated by Smith, D.E., Latham, M.L. Open Court Publishing Company, Chicago, 1925.

[3] Dummit, D.S., Foote, R.M.: Abstract Algebra, 3rd edn. Wiley, New York, 2004, pp. 270–271. ISBN: 978-0-471-43334-7.

[4] Gelfond, A.O.: Transcendental and Algebraic Numbers. Dover Publications, New York

[5] Karpenkov, O.N.: Geometry of Continued Fractions. Springer, Cham, 2022. https://doi.org/10.1007/978-3-030-95266-4.

[6] Khinchin, A.Ya.: Continued Fractions, 3rd edn. University of Chicago Press, Chicago, IL 1964. Translated from the Russian by Scripta Technica, Inc.

[7] Knuth, D.E.: The Art of Computer Programming, Volume 2: Seminumerical Algorithms, 3rd edn. Addison–Wesley, Boston,1997. ISBN: 0-201-89684-2.
[8] Wadhawan, N.K.: Scientific Calculator with the Aid of Geometry and Based upon it a Mechanical Calculator. In: Science and Technology Journal, vol. 11, no. 2, pp. 1–13. 2023. [DOI: 10.22232/stj.2023.11.02.02.